J. Fluid Mech. (1975), *vol.* 70, *part* 2, *pp.* 287–304
Printed in Great Britain

Internal gravity waves: critical layer absorption in a rotating fluid

By R. GRIMSHAW

Mathematics Department, University of Melbourne,
Parkville, Victoria 3052, Australia

(Received 16 September 1974)

The propagation of internal gravity waves in a shear flow in a rotating fluid is examined for the case when the rotation vector is inclined to the vertical. It is shown that internal gravity waves approaching a critical level, where ω^*, the Doppler-shifted frequency, equals $2\Omega_V$, the vertical component of the Coriolis parameter, will be either transmitted or absorbed according as

$$W_g \omega^* 2\Omega_V \{m(U_z + 2\Omega_H) - lV_z\} \lesseqgtr 0;$$

here W_g is the vertical group velocity, $2\Omega_H$ is the horizontal component of the Coriolis parameter, l and m are the easterly and northerly wavenumber components, and U_z and V_z are the shear rates of the easterly and northerly components of the mean flow. Between critical levels, wave action flux is conserved. However, for a wave absorbed at a critical level, the wave action flux is attenuated by a factor $\exp\{-2\pi|[m(U_z+2\Omega_H)-lV_z]/(lU_z+mV_z)|\}$. The phenomenon is also analysed using a WKBJ approximation.

1. Introduction

It is well known that internal gravity waves propagating in a shear flow can be absorbed at certain critical levels, at which the local vertical wavenumber is infinite and the local vertical group velocity is zero. The phenomenon was identified by Bretherton (1966), using a WKBJ approximation and the related concept of a wave packet. Subsequently, a more sophisticated analysis (Booker & Bretherton 1967) showed that an internal gravity wave may be transmitted through a critical level, but is heavily attenuated for Richardson numbers of order unity, or greater. Recently, it has been shown that hydromagnetic waves possess critical levels in a variety of contexts, involving the presence of density stratification, rotation, a shear flow or combinations of these (see Acheson 1972, 1973; McKenzie 1973; Rudraiah & Venkatachalappa 1972*a*, *b*).

For internal gravity waves in a shear flow, the critical level is the height at which the Doppler-shifted wave frequency is zero. Booker & Bretherton (1967) showed that the vertical transport of horizontal momentum by internal gravity waves is independent of height except at critical levels, where there is a discontinuity and the waves are attenuated by a factor $\exp\{-2\pi(R_c-\frac{1}{4})^{\frac{1}{2}}\}$; here R_c is the Richardson number at the critical level. The local Richardson number R is

$$R = N^2/(U_z^2 + V_z^2), \tag{1.1}$$

where N^2 is the Brunt–Väisälä frequency and $(U(z), V(z), 0)$ is the mean horizontal velocity. It was assumed that R was everywhere greater than $\frac{1}{4}$, as this is a sufficient condition for stability of the mean flow.

Jones (1967) examined the effects of rotation about a vertical axis. He showed that the single critical level is replaced by the two levels at which the Doppler-shifted wave frequency is equal in magnitude to the Coriolis frequency. In addition, the vertical flux of horizontal momentum is not conserved, but the vertical flux of angular momentum (about the vertical axis) is conserved and is a suitable measure of wave intensity. He was then able to demonstrate that, if the Coriolis frequency is much smaller than the Brunt–Väisälä frequency, the wave attenuation process away from the critical levels is still described accurately by Booker & Bretherton's results, although the behaviour of the waves in the vicinity of the critical levels is substantially altered.

This paper will examine the effects of rotation about an axis inclined to the vertical. It will be shown that there are again two critical levels at which the Doppler-shifted wave frequency is equal in magnitude to $2\Omega_V$, the *vertical component* of the Coriolis parameter. However, the effect of $2\Omega_H$, the *horizontal component* of the Coriolis parameter, is to cause each critical level to act as a valve. For other examples of the valve effect at a critical level, see Acheson (1973) and McKenzie (1973). A wave of Doppler-shifted frequency ω^* and horizontal wavenumber components l and m (easterly and northerly components respectively) will pass through the critical level without attenuation if its vertical propagation speed W_g is such that

$$W_g \omega^* 2\Omega_V \{m(U_z + 2\Omega_H) - lV_z\} < 0; \tag{1.2}$$

a wave satisfying the opposite inequality will be attenuated at the critical level by a factor

$$\exp\left\{-2\pi \left| \frac{m(U_z + 2\Omega_H) - lV_z}{lU_z + mV_z} \right| \right\}. \tag{1.3}$$

Here $U(z)$ and $V(z)$ are the easterly and northerly components of the mean flow respectively, and the shear rates U_z and V_z are evaluated at the critical level. In addition, it will be shown that a suitable measure of wave intensity is the wave action density $\mathscr{F} = \mathscr{E}/\omega^*$, which is conserved between critical levels; here $\mathscr{E}$ is the wave energy density. However, if the Coriolis parameter is much smaller than the Brunt–Väisälä frequency, the *net* wave attenuation away from the critical levels suffered by a wave passing through both critical levels is still accurately described by Booker & Bretherton's results.

In § 2 the linearized equations of motion for the perturbed state are formulated in the Boussinesq approximation. In § 3 the solutions of the equations in the vicinity of each critical level are obtained by the method of Frobenius, and interpreted by the principle of conservation of wave action. In § 4 a WKBJ approximation is used to supplement the critical level analysis. The appendix contains a derivation of the WKBJ approximation.

2. Equations of motion

Let L be a length scale characteristic of both the perturbed and mean flows and let N_1^{-1} be a time scale, where N_1 is a typical value of the Brunt–Väisälä frequency. Then

$$\epsilon = N_1^2 L/g \tag{2.1}$$

is a small parameter, characterizing the magnitude of the density perturbations. If c_1 is a typical value of the speed of sound, then

$$F = gL/c_1^2 \tag{2.2}$$

is also a small parameter, being the ratio of L to the 'scale height' of the atmosphere, and indicative of the effects of compressibility. The ratio F/ϵ is a property of the mean state; for an ideal gas $F/\epsilon = \gamma - 1/\gamma$, where γ is the ratio of the specific heats; it will be assumed that F is $O(\epsilon)$. Let the velocity scale be $N_1 L$, the density scale be ρ_1 and the pressure scale $\rho_1 gL$. Then the equations governing conservation of mass, momentum and entropy for an inviscid non-diffusive fluid referred to a frame rotating with constant angular velocity $\mathbf{\Omega}$ are, respectively, using non-dimensional variables,

$$d\rho/dt + \rho\nabla . \mathbf{u} = 0, \tag{2.3}$$

$$\frac{d\mathbf{u}}{dt} + 2\mathbf{\Omega} \times \mathbf{v} + \epsilon^{-1}\frac{\nabla p}{\rho} + \epsilon^{-1}\mathbf{k} = 0, \tag{2.4}$$

$$\frac{d\rho}{dt} - F\frac{1}{c^2}\frac{dp}{dt} = 0. \tag{2.5}$$

Here $\mathbf{u}$ is the velocity relative to the rotating frame, p is the pressure, ρ is the density, c is the speed of sound (a prescribed function of p and ρ) and $\mathbf{k}$ is a unit vector in the vertical direction. The centrifugal effects of rotation have been absorbed into the gravitational term $g\mathbf{k}$, which is assumed to be a constant. The x and y axes will be in the easterly and northerly directions respectively, and the z axis will be in the vertical direction. Thus

$$2\mathbf{\Omega} = (0, 2\Omega_H, 2\Omega_V). \tag{2.6}$$

Note that $2\Omega_H$ and $2\Omega_V$ are the non-dimensional horizontal and vertical components of the Coriolis parameter, and have been scaled by N_1.

Let the mean flow be represented by a velocity $\mathbf{V}$, pressure Q and density R, satisfying (2.3)–(2.5). It will be assumed that the mean flow is time independent, and that $\mathbf{V}$ is horizontal and a function of z only;

$$\mathbf{V} = (U(z), V(z), 0). \tag{2.7}$$

It follows that

$$\epsilon R 2\mathbf{\Omega} \times \mathbf{V} + \nabla Q + R\mathbf{k} = 0, \tag{2.8}$$

$$\mathbf{V}.\nabla R = 0. \tag{2.9}$$

Thus the mean flow is in hydrostatic and geostrophic balance. To leading order in ϵ, there is just hydrostatic balance, and consequently the horizontal gradients of Q are smaller by a factor $O(\epsilon)$ than the vertical gradient. Elimination of Q from (2.8) gives

$$R_x = -\epsilon 2\mathbf{\Omega}.\nabla(RV), \quad R_y = \epsilon 2\mathbf{\Omega}.\nabla(RU). \tag{2.10}$$

If this is to be consistent with (2.9) then the ratio V/U must be a constant: the mean velocity has a *constant direction*. The general solution of (2.10) shows that

$$RU = f\left(\int^z \frac{dz}{U} + \epsilon\left\{2\Omega_V\left[y - \frac{Vx}{U}\right] - 2\Omega_H z\right\}\right), \tag{2.11}$$

where f is an arbitrary function. The Brunt–Väisälä frequency is N, where

$$\epsilon N^2 = -(R_z/R + F/C^2); \tag{2.12}$$

here C is the speed of sound c as a function of (Q, R). The presence of ϵ in (2.12) is necessitated by the definition of ϵ in (2.1) and incorporates the approximation that the density gradient of the mean flow is small ($O(\epsilon)$).

The equations governing the perturbed motion are obtained by linearizing (2.3)–(2.5) about the mean flow solution. In this and the subsequent section, solutions of the perturbed equations will be sought using separation of variables. This implies, in particular, that N^2, R_x/R and R_y/R must be functions of z only. It is clear from (2.11) that this will be possible only in the Boussinesq approximation, in which ϵ, $F \to 0$ (in the perturbed equations). Also, it follows from (2.10) that

$$R_x/R = -\epsilon 2\Omega_V V_z + O(\epsilon^2), \quad R_y/R = \epsilon 2\Omega_V U_z + O(\epsilon^2). \tag{2.13}$$

Hence in order to separate variables it will be assumed that V_z and U_z are *both constants*. It follows that

$$\mathbf{V} = \boldsymbol{\alpha} z, \quad \boldsymbol{\alpha} = (\alpha, \beta, 0), \tag{2.14}$$

where α and β are the constant shear rates U_z and V_z respectively. There is no loss of generality in supposing that $\mathbf{V}$ vanishes at $z = 0$, as this can always be achieved by a suitable choice of origin for z and use of the condition that $\mathbf{V}$ has a constant direction. To obtain the perturbed equations, let

$$\mathbf{u} = \mathbf{V} + \hat{\mathbf{v}}, \quad p = Q + \epsilon\hat{q}, \quad \rho = R(1 + \epsilon\hat{r}). \tag{2.15}$$

Substitution into (2.3)–(2.5), linearization in the perturbed variables $\mathbf{v}$, $\hat{q}$ and $\hat{r}$, and application of the Boussinesq approximation ($\epsilon, F \to 0$) lead to

$$\nabla . \hat{\mathbf{v}} = 0, \tag{2.16}$$

$$\frac{D\hat{\mathbf{v}}}{Dt} + \hat{w}\boldsymbol{\alpha} + 2\boldsymbol{\Omega} \times \hat{\mathbf{v}} + \frac{1}{R}\nabla\hat{q} + \hat{r}\mathbf{k} = 0, \tag{2.17}$$

$$D\hat{r}/Dt - \hat{u}2\Omega_V\beta + \hat{v}2\Omega_V\alpha - N^2\hat{w} = 0, \tag{2.18}$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \alpha z\frac{\partial}{\partial x} + \beta z\frac{\partial}{\partial y}. \tag{2.19}$$

Here $(\hat{u}, \hat{v}, \hat{w})$ are the components of the perturbed velocity $\hat{\mathbf{v}}$. Note that the presence of ϵ in (2.15) is required by the Boussinesq approximation, which has been anticipated in the definition of ϵ in (2.1).

There does not seem to be any sufficient condition known that will ensure the stability of the mean flow considered here. In the absence of rotation, a sufficient condition for stability is that the Richardson number $N^2/\alpha^2 + \beta^2$ be greater than $\frac{1}{4}$; it seems plausible that in the presence of rotation a sufficiently high value of the Richardson number will ensure stability. It will be assumed here that the mean flow is stable, and consequently the perturbed equations possess wavelike solutions, for which

$$\left.\begin{aligned} \hat{\mathbf{v}} &= \mathrm{Re}\{\mathbf{v}(z)\exp(ilx+imy-i\omega t)\},\\ \hat{q} &= \mathrm{Re}\{q(z)\exp(ilx+imy-i\omega t)\},\\ \hat{r} &= \mathrm{Re}\{r(z)\exp(ilx+imy-i\omega t)\}.\end{aligned}\right\} \tag{2.20}$$

The resulting equations are

$$ilu+imv+w_z = 0, \tag{2.21}$$

$$-i\omega^*\mathbf{v}+w\boldsymbol{\alpha}+2\boldsymbol{\Omega}\times\mathbf{v}+i\boldsymbol{\kappa}_H q/R+(q_z/R+r)\,\mathbf{k} = 0, \tag{2.22}$$

$$-i\omega^* r-u2\Omega_V\beta+v2\Omega_V\alpha-N^2w = 0, \tag{2.23}$$

where

$$\left.\begin{aligned} \omega^* &= \omega-\boldsymbol{\kappa}_H.\mathbf{V} = \omega-(\boldsymbol{\kappa}_H.\boldsymbol{\alpha})\,z,\\ \boldsymbol{\kappa}_H &= (l,m,0).\end{aligned}\right\} \tag{2.24}$$

ω^* is the Doppler-shifted, or intrinsic, frequency and $\boldsymbol{\kappa}_H$ is the horizontal wavenumber vector. Finally, elimination of all variables except w leads to the equation

$$\{(\omega^{*2}-(2\Omega_V)^2)\}\,w_{zz}+\{-2(2\Omega_V)^2\sigma/\omega^*-2i(2\Omega_V)\,\mu\}\,w_z$$
$$+\{-2i(2\Omega_V)\,\sigma\mu/\omega^*+(N^2-\omega^{*2})\,(l^2+m^2)+2\Omega_H m\mu\}\,w = 0, \tag{2.25}$$

where

$$\mu = [m(2\Omega_H+\alpha)-l\beta],\quad \sigma = \boldsymbol{\kappa}_H.\boldsymbol{\alpha}. \qquad (2.26),\ (2.27)$$

This equation agrees with that obtained by Jones (1967) for the case $2\Omega_H = 0$, $\beta = 0$. The pressure perturbation is

$$q/R = \frac{i[\omega^{*2}-(2\Omega_V)^2]\,w_z+[i\omega^*(\sigma+2\Omega_H m)+2\Omega_V\mu]\,w}{\omega^*(l^2+m^2)}. \tag{2.28}$$

3. Critical levels

The equation (2.25) for w has singularities at the critical levels, where

$$\omega^{*2} = (2\Omega_V)^2.$$

There is also an apparent singularity where $\omega^* = 0$, but it will be shown below that solutions of (2.25) are always regular at $\omega^* = 0$. Before considering the solutions of (2.25) near the critical levels, a useful invariant of (2.25), the wave action flux, will be obtained.

If (2.22) is multiplied by $\bar{\mathbf{v}}$ (the complex conjugate of $\mathbf{v}$), $\boldsymbol{\kappa}_H.\bar{\mathbf{v}}$ eliminated by use of (2.21) and r eliminated by use of (2.23), it follows that

$$\frac{1}{R}\frac{\partial}{\partial z}(q\bar{w})+w\boldsymbol{\alpha}.\bar{\mathbf{v}}-\frac{\bar{w}}{i\omega^*}\boldsymbol{\alpha}.2\boldsymbol{\Omega}\times\mathbf{v}$$
$$-i\omega^*|\mathbf{v}|^2-\frac{N^2|w|^2}{i\omega^*}+\frac{2\Omega_H\alpha|w|^2}{i\omega^*}+2\boldsymbol{\Omega}.\mathbf{v}\times\bar{\mathbf{v}} = 0. \tag{3.1}$$

Further, elimination of the horizontal component of $\mathbf{v}$ by use of (2.22), and use of the fact that

$$\sigma = \boldsymbol{\kappa}_H.\boldsymbol{\alpha} = -\partial\omega^*/\partial z \tag{3.2}$$

leads to the equation

$$\frac{\omega^*}{R}\frac{\partial}{\partial z}\left(\frac{q\bar{w}}{\omega^*}\right) - i\omega^*|\mathbf{v}|^2 - \frac{N^2|w|^2}{i\omega^*} + \frac{|w|^2(\alpha^2+\beta^2)}{i\omega^*} + \frac{2\Omega_H\alpha|w|^2}{i\omega^*} + 2\boldsymbol{\Omega}.\mathbf{v}\times\bar{\mathbf{v}} + (w\boldsymbol{\alpha}.\bar{\mathbf{v}} - \bar{w}\boldsymbol{\alpha}.\mathbf{v}) = 0. \tag{3.3}$$

Taking the real part of this equation, it follows that

$$\partial\mathscr{G}/\partial z = 0,$$

where

$$\mathscr{G} = \tfrac{1}{2}\,\mathrm{Re}\,[q\bar{w}/\omega^*]. \tag{3.4}$$

From (2.28),

$$\mathscr{G}/R = \tfrac{1}{2}\,\mathrm{Re}\left[\frac{i(\omega^{*2}-(2\Omega_V)^2)\,\bar{w}w_z + 2\Omega_V\mu|w|^2}{\omega^{*2}(l^2+m^2)}\right]. \tag{3.5}$$

Of course, this result does not hold at a critical level. Equation (3.4) shows that $\mathscr{G}$ is a constant between critical levels, but suffers a discontinuity at each critical level. $\mathscr{G}$ is the wave action flux. In the absence of rotation, $\mathscr{G}$ is proportional to the vertical flux of horizontal momentum (cf. Booker & Bretherton 1967); if $2\Omega_H = 0$, $\mathscr{G}$ may be related to the vertical flux of angular momentum (Jones 1967); in the present case, it would seem that $\mathscr{G}$ has no simple interpretation in terms of flux of linear or angular momentum. However, $\frac{1}{2}\,\mathrm{Re}\,[q\bar{w}]$ is the vertical flux of energy, and hence a vertical propagation velocity W_g can be defined such that

$$\mathscr{E}W_g = \tfrac{1}{2}\,\mathrm{Re}\,[q\bar{w}], \tag{3.6}$$

where $\mathscr{E}$ is the local energy density (the time average of $\frac{1}{2}R|\hat{\mathbf{v}}|^2 + \frac{1}{2}R|\hat{r}|^2/N^2$, or $\frac{1}{4}R|\mathbf{v}|^2 + \frac{1}{4}R|r|^2/N^2$). The wave action density is $\mathscr{F}$, where $\mathscr{G} = W_g\mathscr{F}$. The conservation of wave action for a shear flow in the absence of rotation was observed by Booker & Bretherton (1967); its validity under the WKBJ approximation has been established in a variety of contexts (Bretherton & Garrett 1968).

3.1. *Singularity at* $\omega^* = 2\Omega_V$

Equation (2.25) has a regular singularity at z_+, where

$$z_+ = (\omega - 2\Omega_V)/\sigma. \tag{3.7}$$

The method of Frobenius shows that w can be written in the form

$$w = (z-z_+)^{\lambda}\,(1 + a_1(z-z_+) + \ldots), \tag{3.8}$$

where the power series in $z-z_+$ will converge for $|z-z_+| < |2\Omega_V/\sigma|$. The indicial equation for λ has solutions

$$\lambda = 0,\ -i\mu/\sigma. \tag{3.9}$$

(i) $\lambda = -i\mu/\sigma$. To determine the appropriate branch for $(z-z_+)^{\lambda}$ in (3.8), it will be assumed that $\omega = \omega_R + i\omega_I$, with $\omega_I > 0$; this 'radiation condition' arises from causality considerations (cf. Booker & Bretherton 1967). Thus $\mathrm{Im}\,(z_+) \gtrless 0$ according as $\sigma \gtrless 0$; in the limit $\omega_I \to 0$, the appropriate path for determining the branch of $(z-z_+)^{\lambda}$ passes below (above) z_+. Hence

$$(z-z_+)^{-i\mu/\sigma} = \begin{cases} |z-z_+|^{-i\mu/\sigma} & \text{for}\quad z > z_+, \\ |z-z_+|^{-i\mu/\sigma}\exp\left(-\pi\mu/|\sigma|\right) & \text{for}\quad z < z_+. \end{cases} \tag{3.10}$$

Substitution of this solution into (3.5) shows that

$$\mathscr{G}/R = \left\{\begin{array}{ll} -\tfrac{1}{2}\mu/2\Omega_V(l^2+m^2) & \text{for} \quad z > z_+, \\ -\tfrac{1}{2}\mu \exp(-2\pi\mu/|\sigma|)/2\Omega_V(l^2+m^2) & \text{for} \quad z < z_+. \end{array}\right\} \tag{3.11}$$

Recalling that the sign of $\omega^*\mathscr{G}$ (here $2\Omega_V\mathscr{G}$) determines the direction of the wave, it follows that if $\mu > 0\,(< 0)$ the solution represents a wave propagating vertically downwards (upwards); in either case the wave energy flux is attenuated across the critical level by the factor

$$\exp(-2\pi|\mu/\sigma|). \tag{3.12}$$

Note that this solution is characterized by

$$W_g\mu\omega^*2\Omega_V < 0, \tag{3.13}$$

where W_g is the vertical propagation velocity (3.6).

(ii) $\lambda = 0$. This solution is regular at z_+. Evaluating (3.5) at $z = z_+$ shows that

$$\mathscr{G}/R = \tfrac{1}{2}\mu/2\Omega_V(l^2+m^2). \tag{3.14}$$

Hence this solution represents a wave propagating vertically upwards or downwards according as $\mu \gtrless 0$. This solution is characterized by

$$W_g\mu\omega^*2\Omega_V > 0. \tag{3.15}$$

Thus the critical level at $\omega^* = 2\Omega_V$ acts as a valve. A wave propagating towards the critical level will be transmitted unattenuated or absorbed [with attenuation factor (3.12)] according as $W_g\mu\omega^*2\Omega_V \gtrless 0$. Acheson (1973) has identified a similar valve effect in a variety of other contexts.

3.2. *Singularity at* $\omega^* = -2\Omega_V$

There is a regular singularity at z_-, where

$$z_- = (\omega + 2\Omega_V)/\sigma. \tag{3.16}$$

Seeking a solution of the form (3.8) in the variable $z - z_-$ leads to an indicial equation for λ with solutions

$$\lambda = 0, i\mu/\sigma. \tag{3.17}$$

An analysis similar to that for $\omega^* = 2\Omega_V$ shows that, for the solution $\lambda = i\mu/\sigma$, $\mathscr{G}$ is given by

$$\mathscr{G}/R = \left\{\begin{array}{ll} -\tfrac{1}{2}\mu/2\Omega_V(l^2+m^2) & \text{for} \quad z > z_-, \\ -\tfrac{1}{2}\mu \exp(2\pi\mu/|\sigma|)/2\Omega_V(l^2+m^2) & \text{for} \quad z < z_-. \end{array}\right\} \tag{3.18}$$

Hence, this solution represents a wave propagating vertically upwards or downwards according as $\mu \gtrless 0$; in either case the wave energy flux is attenuated by the factor (3.12). This solution is again characterized by (3.13). For the solution given by $\lambda = 0$, $\mathscr{G}$ is again given by (3.14), and hence represents a wave propagating vertically downwards or upwards according as $\mu \gtrless 0$; this solution is again characterized by (3.15).

Thus the critical level at $\omega^* = -2\Omega_V$ also acts as a valve, operating on the same criterion as the critical level at $\omega^* = 2\Omega_V$; a wave propagating towards a critical level is transmitted or absorbed according as $W_g \mu\omega^* 2\Omega_V \gtrless 0$.

3.3. *Singularity at* $\omega^* = 0$

There is a regular singularity at z_0, where

$$z_0 = \omega/\sigma. \tag{3.19}$$

Seeking a solution of the form (3.8) in the variable $z-z_0$ leads to an indicial equation for λ with solutions

$$\lambda = 0, 3. \tag{3.20}$$

The general solution is thus regular and has the form

$$w = a_0\{1-i\mu(z-z_0)/2\Omega_V+\ldots\}, \tag{3.21}$$

where a_0 is an arbitrary constant. Substitution into (3.5) shows that $\mathscr{G}$ is a regular function of z near z_0 and is continuous at z_0.

3.4. $|2\Omega_V| \ll |\omega^*| \ll N^2$

A wave transmitted by one critical level, and thus satisfying (3.15), will *either continue* to propagate in the same direction and thus be absorbed at the other critical level [where (3.13) holds], *or* will be reflected at some height between the critical levels, will then satisfy (3.13) and so will be absorbed at the critical level from which it was originally transmitted, albeit from the opposite side. From arguments given below, substantiated by a WKBJ approximation in §4, the latter alternative is more likely. As $2\Omega_H, 2\Omega_V \to 0$ both critical levels coincide with $\omega^* = 0$, and the valve effect disappears.

In order to examine the relationship of these results to those when $2\Omega_V$ and $2\Omega_H$ are zero, it will now be supposed that $|\omega^*| \gg |2\Omega_V|$, but that $|\omega^*| \ll N$, where $N \gg |2\Omega_V|, |2\Omega_H|$. Then (2.25) is approximated by

$$\begin{aligned}\{(\omega^{*2}-(2\Omega_V)^2\}\,w_{zz}+\{-2(2\Omega_V)\,2\sigma/\omega^*-2i(2\Omega_V)\,\mu\}\,w_z&\\ +\{-2i(2\Omega_V)\,\sigma\mu/\omega^*+N^2(l^2+m^2)+2\Omega_H m\mu\}\,w = 0.&\end{aligned} \tag{3.22}$$

A solution to this approximate equation is sought of the form

$$(z-z_0)^\lambda\{1+a_1/(z-z_0)+\ldots\}, \tag{3.23}$$

where the power series in $(z-z_0)^{-1}$ will converge for $|z-z_0| > |2\Omega_V|$. This is the device used by Jones (1967) in the case $2\Omega_H = 0, V_z = 0$. The indicial equation for λ has solutions

$$\lambda = \tfrac{1}{2} \pm i\nu,$$

where
$$\nu = \{[N_0^2(l^2+m^2)+2\Omega_H m\mu]/\sigma^2-\tfrac{1}{4}\}^{\frac{1}{2}}; \tag{3.24}$$

also
$$a_1 = -i2\Omega_V\,\mu/\sigma^2.$$

It will be assumed that ν is real and positive (noting that this is a sufficient condition for stability if $2\Omega_H = 2\Omega_V = 0$). It can be shown (by standard iteration arguments) that solutions of (3.22) will approximate to solutions of (2.25) if $|\omega^*| \ll N$. Also the solutions of (3.22) are approximately $(z-z_0)^\lambda$ [λ given by (3.24)] if $|z-z_0| \ll a_1$. Thus the general solution of (2.25) is approximately

$$w \approx A(z-z_0)^{\frac{1}{2}+i\nu}+B(z-z_0)^{\frac{1}{2}-i\nu}, \tag{3.25}$$

provided that

$$|2\Omega_V \mu/\sigma^2| \ll |z-z_0| \ll N/|\sigma|. \tag{3.26}$$

Here A and B are arbitrary complex constants. The error in (3.25) is

$$O(2\Omega_V \mu/\sigma^2 |z-z_0|,\ |z-z_0|^2 \sigma^2/N^2).$$

The appropriate branch for $(z-z_0)^\lambda$ is determined in a manner similar to that in §3.1 above. Substitution into (3.5) then shows that, if (3.26) holds,

$$\mathscr{G}/R \approx \left\{ \begin{array}{ll} -\tfrac{1}{2}\nu\{|A|^2-|B|^2\}/l^2+m^2 & \text{for}\quad z-z_0 > 0, \\ \tfrac{1}{2}\nu\{|A|^2\exp(2\pi\nu\,\mathrm{sgn}\,\sigma)-|B|^2\exp(-2\pi\nu\,\mathrm{sgn}\,\sigma)\}/l^2+m^2 & \text{for}\quad z-z_0 < 0. \end{array} \right\} \tag{3.27}$$

The error here is $O(|2\Omega_V/N|^{\frac{2}{3}})$. Recalling that $\omega^* = \sigma(z_0-z)$, it may be shown that the solution corresponding to the constant A is propagating vertically upwards or downwards according as $\sigma \gtrless 0$. The solution corresponding to the constant B is propagating vertically downwards or upwards according as $\sigma \gtrless 0$. In either case a wave propagating towards the critical level is attenuated by a factor

$$\exp(-2\pi\nu). \tag{3.28}$$

For $N \gg |2\Omega_V|, |2\Omega_H|$ this agrees with the result obtained by Booker & Bretherton (1967), and shows that away from the critical levels the net attenuation is unaffected by the rotation. However, the preceding analysis clearly shows that rotation alters the behaviour near the critical levels in a radical way. Also the net attenuation described by (3.28) is much larger than that described by (3.12) (if $N \gg |2\Omega_V|, |2\Omega_H|$ and the Richardson number is large); this must be accounted for by a substantial change in the amplitude of w, that is, in the constants A and B in (3.25) and $a_\pm$ and $b_\pm$ in (3.29), as the critical levels are approached.

It remains to try and connect the solution far from the critical level, given by (3.25) and (3.26), with the solution near the critical levels,

$$w \approx a_\pm(z-z_\pm)^{\mp i\mu/\sigma}+b_\pm \quad \text{for} \quad \omega^* \approx \pm 2\Omega_V. \tag{3.29}$$

To be definite, let $z_+ < z_0 < z_-$ (figure 1) and $\sigma > 0$ (so that $2\Omega_V > 0$). (The other cases are similar and will not be discussed.) Then substitution of (3.29) into (3.5) shows that

$$\left. \begin{array}{ll} \mathscr{G}/R = \dfrac{-\frac{1}{2}\mu}{2\Omega_V(l^2+m^2)}\{|a_+|^2\exp(-2\pi\mu/|\sigma|)-|b_+|^2\} & \text{for}\quad z < z_+, \\ \mathscr{G}/R = \left\{ \begin{array}{l} -\dfrac{\frac{1}{2}\mu}{2\Omega_V(l^2+m^2)}\{|a_+|^2-|b_+|^2\} \\ -\dfrac{\frac{1}{2}\mu}{2\Omega_V(l^2+m^2)}\{|a_-|^2\exp(2\pi\mu/|\sigma|)-|b_-|^2\} \end{array} \right\} & \text{for}\quad z_+ < z < z_-, \\ \mathscr{G}/R = \dfrac{-\frac{1}{2}\mu}{2\Omega_V(l^2+m^2)}\{|a_-|^2-|b_-|^2\} & \text{for}\quad z > z_-. \end{array} \right\} \tag{3.30}$$

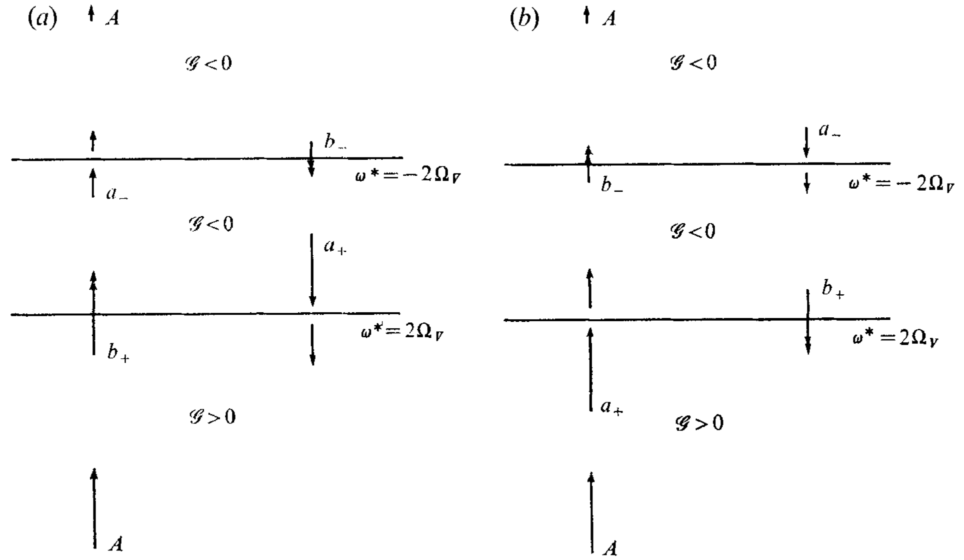

FIGURE 1. (a) $\mu > 0$, (b) $\mu < 0$. Illustrates the wave absorption process for the case σ, $2\Omega_V$ positive. The arrows give a qualitative indication of the respective wave amplitudes. ↠, transmitted wave; →, captured wave.

Also, for $z > z_-$ and $z < z_+$, $\mathscr{G}$ is approximately given by (3.27) in terms of A and B. Suppose that the solution is an upward-propagating wave for $z < z_+$, so that $B = 0$ (as $\sigma > 0$). Then $\omega^* \mathscr{G}$ is positive for $z < z_+$, and from (3.27) is also positive for $z > z_-$. From (3.30) it follows that

$$\left.\begin{aligned} \mu\{|a_+|^2 \exp(-2\pi\mu/|\sigma|) - |b_+|^2\} &< 0, \\ \mu\{|a_-|^2 - |b_-|^2\} &> 0. \end{aligned}\right\} \tag{3.31}$$

Substituting (3.31) into (3.30) shows that $\mathscr{G} < 0$ for $z_+ < z < z_-$, and hence that

$$\mu\{|a_+|^2 - |b_+|^2\} > 0. \tag{3.32}$$

Combining (3.31) and (3.32), it follows that

$$\mu|a_+|^2 \exp(-2\pi\mu/|\sigma|) < \mu|b_+|^2 < \mu|a_+|^2. \tag{3.33}$$

Thus a wave which is upward propagating far from the critical levels becomes near $\omega^* = 2\Omega_V$ a wave with both upward and downward components, although still carrying wave energy *upwards*. Just above $\omega^* = 2\Omega_V$, the wave again has both upward and downward components, and is carrying wave energy *downwards*. If $\mu < 0$, the dominant component below $\omega^* = 2\Omega_V$ corresponds to a_+, and is propagating upwards and absorbed at this level; just above $\omega^* = 2\Omega_V$ the dominant component corresponds to b_+, and is propagating downwards and is transmitted at this level. If $\mu > 0$, the dominant component below $\omega^* = 2\Omega_V$ corresponds to b_+, and is propagating upwards and transmitted at this level; however, just above $\omega^* = 2\Omega_V$ the dominant component corresponds to a_+, and is propagating downwards and is absorbed at this level. This suggests that the upward wave is reflected somewhere between the critical levels, a result substantiated by the WKBJ analysis of §4. In either case the principal absorption takes place at $\omega^* = 2\Omega_V$, the first critical level encountered. The situation is described diagrammatically in figure 1. Near $\omega^* = -2\Omega_V$, the wave is again carrying wave energy upwards, although diminished in amplitude.

Finally, the case $\mu = 0$ will be briefly considered. Near the critical level $\omega^* = 2\Omega_V$,

$$w \approx a_+ \log (z - z_+) + b_+, \tag{3.34}$$

and it may be shown that

$$\mathscr{G}/R = \left\{ \begin{array}{ll} -\tfrac{1}{2}\sigma\,\mathrm{Re}\,(ia_+\bar{b}_+)/2\Omega_V(l^2+m^2) & \text{for} \quad z > z_+, \\ \{\tfrac{1}{2}\pi|\sigma|\;|a_+|^2 - \tfrac{1}{2}\sigma\,\mathrm{Re}\,(ia_+\bar{b}_+)\}/2\Omega_V(l^2+m^2) & \text{for} \quad z < z_-. \end{array} \right\} \tag{3.35}$$

The a_+ wave (for which $b_+ = 0$) represents a wave propagating vertically upwards, whose wave energy flux is reduced by an amount $\tfrac{1}{2}R\pi|\sigma|\;|a_+|^2(l^2+m^2)^{-1}$ at the critical level. The b_+ wave (for which $a_+ = 0$) has zero wave energy flux.

4. WKBJ approximation

When the mean flow varies only slightly over distances of the order of a wavelength, the WKBJ approximation and the concept of a wave packet provide a useful alternative approach to critical level behaviour. The perturbed motion is described by plane waves with a local frequency ω, wavenumber $\boldsymbol{\kappa}$ and amplitude a, which may vary with position and time on a scale provided by the mean flow. The plane waves are assumed to be modulated such that the amplitude is very small except within a certain region, which moves with the group velocity. This approach to critical level behaviour was initiated by Bretherton (1966) for internal gravity waves in the absence of rotation.

The WKBJ approximation is derived in the appendix for a general mean flow. For the particular mean flow considered in §§2 and 3, $\mathbf{V}$ will be a function of $Z = \epsilon z$, where ϵ [see (2.1)] now measures the ratio of a wavelength to the length scale of the mean flow. From (2.14) it follows that

$$\mathbf{V} = \mathbf{A}Z, \quad \mathbf{A} = (A, B, 0) = \epsilon^{-1}\boldsymbol{\alpha}. \tag{4.1}$$

Here the requirements of the WKBJ approximation imply that the shear rate $\boldsymbol{\alpha}$ is $O(\epsilon)$, so that $\mathbf{A}$ is $O(1)$. The Richardson number is now $O(\epsilon^{-2})$. The perturbed motion is now locally a plane wave given by (A 9), (A 10) (see appendix) whose frequency ω and wavenumber $\boldsymbol{\kappa}$ [see (A 11)] satisfy the dispersion relation (A 14):

$$\omega^{*2}(l^2+m^2+n^2) = (2\Omega_H m + 2\Omega_V n)^2 + N^2(l^2+m^2). \tag{4.2}$$

Here (l, m, n) are the components of the wavenumber $\boldsymbol{\kappa}$ and ω^* is the Doppler-shifted, or intrinsic, frequency,

$$\left. \begin{array}{c} \omega^* = \omega - \boldsymbol{\kappa}.\mathbf{V} = \omega - SZ, \\ S = Al + Bm = \boldsymbol{\kappa}.\mathbf{A}. \end{array} \right\} \tag{4.3}$$

Note that σ [see (2.27)] is ϵS, and is $O(\epsilon)$, consistent with WKBJ approximation. Equation (4.2) is a partial differential equation for the phase of the wave [see

A 11)]; here, the appropriate solution is that with ω, l and m constants, and then (4.2) determines n as a function of Z. The wave is confined to those values of Z for which (4.2) has real solutions for n, i.e. those values of Z for which

$$(l^2+m^2)(N^2-\omega^{*2})[\omega^{*2}-(2\Omega_V)^2]+(2\Omega_H)^2 m^2\omega^{*2} \geqslant 0. \tag{4.4}$$

The inequality (4.4) shows that

$$\omega^2_{\text{min}} \leqslant \omega^{*2} \leqslant \omega^2_{\text{max}},$$

where

$$\begin{aligned}2\omega^2_{\text{max, min}} = {} & \{N^2+(2\Omega_V)^2+(2\Omega_H)^2 m^2/l^2+m^2\} \\ & \pm[\{N^2+(2\Omega_V)^2+(2\Omega_H)^2 m^2/l^2+m^2\}^2-4N^2(2\Omega_V)^2]^{\frac{1}{2}}.\end{aligned} \tag{4.5}$$

Note that $\omega^2_{\text{min}} \leqslant (2\Omega_V)^2$, with equality only if $2\Omega_H m$ is zero. The vertical component of the group velocity is $W_g = \mathbf{c}.\mathbf{k}$ [see (A 25)], where

$$\omega^*(l^2+m^2+n^2)W_g = (2\Omega_H m+2\Omega_V n)[2\Omega_V(l^2+m^2)-2\Omega_H mn]-N^2n(l^2+m^2). \tag{4.6}$$

A wave packet (identified by particular values of ω, l and m) has critical levels where $n \to \infty$, that is, at those levels where $\omega^{*2} = (2\Omega_V)^2$. From (4.2),

$$\begin{aligned}n(\omega^{*2}-(2\Omega_V)^2) = {} & (2\Omega_H)(2\Omega_V)m \\ & \pm\{(N^2-\omega^{*2})[\omega^{*2}-(2\Omega_V)^2](l^2+m^2)+(2\Omega_H)^2 m^2\omega^{*2}\}^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

Consider first the critical level $\omega^* = 2\Omega_V$. As $\omega^* \to 2\Omega_V$,

$$n \sim n_1 = (2\Omega_H)m/(\omega^*-2\Omega_V), \tag{4.8}$$

or

$$n \sim n_2 = -\{[N^2-(2\Omega_V)^2](l^2+m^2)+(2\Omega_H)^2 m^2\}/2m(2\Omega_H)(2\Omega_V). \tag{4.9}$$

The first root becomes infinite as the critical level is approached, but the second root is finite. The corresponding forms for the group velocity are

$$W_g \sim -(2\Omega_H)m/n_1^2, \quad (2\Omega_H)m/(l^2+m^2+n_2^2). \qquad (4.10), (4.11)$$

Thus the first root (n_1) corresponds to capture as $W \to 0$ as $|Z-Z_+|^2$, where Z_+ ($= \epsilon z_+$) is $\omega-2\Omega_V/S$; as the wave packet approaches the critical level the time $T \to \infty$ as $|Z-Z_+|^{-1}$, and so the wave packet is captured. The wave amplitude, measured by the wave action $\mathscr{F}$, is given by (A 28), or, in the present case,

$$\mathscr{F}_T+(\mathscr{F}W_g)_Z = 0. \tag{4.12}$$

As the critical level is approached, $\mathscr{F}$ varies within the wave packet as $|Z-Z_+|^{-2}$, although the total wave action in the wave packet is conserved (cf. Grimshaw 1972). The amplitude a within the wave packet remains bounded, as may be deduced from (A 23) and (A 29). These results for the wave amplitude compare favourably with those of §3, q.v. (3.10) and (4.20) below. However, the second root corresponds to transmission through the critical level, as W_g is finite there. This wave will be reflected at the level where $\omega^* = \omega^*_{\text{min}}$, reverse its direction and be captured at the critical level, albeit from the opposite side. A wave packet is either captured or transmitted according to the criterion

$$W_g\omega^* 2\Omega_V 2\Omega_H m \left\{\begin{array}{ll} < 0 & \text{for capture,} \\ > 0 & \text{for transmission.}\end{array}\right\} \tag{4.13}$$

Next, consider the critical level $\omega^* = -2\Omega_V$. As $\omega^* \rightarrow -2\Omega_V$,

$$n \sim n_1 = -2\Omega_H m/(\omega^* + 2\Omega_V), \tag{4.14}$$

or

$$n \sim n_2 = -\{[N^2 - (2\Omega_V)^2](l^2+m^2) + (2\Omega_H)^2 m^2\}/2m(2\Omega_H)(2\Omega_V). \tag{4.15}$$

The corresponding group velocities are

$$W_g \sim \frac{2\Omega_H m}{n_1^2}, \quad -\frac{2\Omega_H m}{l^2+m^2+n_2^2}. \qquad (4.16), (4.17)$$

Thus the first root (n_1) becomes infinite, W tends to zero and so this root represents capture; the second root is finite and represents transmission. The criterion for capture, or transmission, is again (4.13).

It may be noted that the criterion for the continued validity of the WKBJ approximation is that the fractional change in wavenumber n over one wavelength should remain $O(\epsilon)$ as the critical level is approached. From (4.8),

$$\delta n \approx \frac{2\Omega_H m}{(\omega^* - 2\Omega_V)^2} S\,\delta Z, \quad \delta Z = \epsilon \delta z, \tag{4.18}$$

and so, putting $\delta z = 2\pi/n$,

$$\delta n/n \approx \epsilon 2\pi S/2\Omega_H m. \tag{4.19}$$

This will remain $O(\epsilon)$, provided S is $O(1)$ (i.e. σ is $O(\epsilon)$) and $2\Omega_H m$ is not zero. Also, the solution obtained in §3, (3.8) and (3.10), may be written in the form

$$w \approx \exp\{-i(\mu/\sigma^{-1})\log(z - z_+)\}. \tag{4.20}$$

This may be interpreted as a wavelike solution, with a vertical wavenumber

$$n \approx -\mu/\sigma(z - z_+),$$

or

$$n \approx \{2\Omega_H m + \epsilon(mA - lB)\}/(\omega^* - 2\Omega_V). \tag{4.21}$$

This differs from (4.8) by a term $O(\epsilon)$. Also (4.20) shows that the magnitude of w remains bounded near the critical level.

These results may now be compared with those of §3. In the present notation, the criteria for capture or transmission, (3.13) or (3.15) respectively, derived without using the WKBJ approximation are

$$W_g \omega^* 2\Omega_V \{2\Omega_H m + \epsilon(mA - lB)\} \lesseqgtr 0. \tag{4.22}$$

These differ from (4.13) only by terms $O(\epsilon)$. Also, the results of §3 show that a captured wave is attenuated by the factor (3.12),

$$\exp[-2\pi|\{2\Omega_H m + \epsilon(mA - lB)\}/\epsilon(lA + mB)|]. \tag{4.23}$$

This is transcendentally small in the WKBJ approximation, and is consequently ignored.

The results of this section have so far presupposed that $2\Omega_H m$ is not zero. If $2\Omega_H m$ is zero, then the valve effect disappears, and

$$n = \pm\left\{\frac{(N^2 - \omega^{*2})(l^2+m^2)}{(\omega^{*2} - (2\Omega_V)^2)}\right\}^{\frac{1}{2}}, \tag{4.24}$$

$$W_g = -\frac{[N^2 - (2\Omega_V)^2](l^2+m^2)\,n}{\omega^*(l^2+m^2+n^2)^2}. \tag{4.25}$$

From (4.5), the wave is now confined to the region

$$(2\Omega_V)^2 \leqslant \omega_*^2 \leqslant N^2. \tag{4.26}$$

As $\omega^* \rightarrow 2\Omega_V$, both roots for n become infinite as $|Z-Z_+|^{-\frac{1}{2}}$, W tends to zero as $|Z-Z_+|^{\frac{3}{2}}$, the time $T \rightarrow \infty$ as $|Z-Z_+|^{-\frac{1}{2}}$, and so the wave packet is captured. The wave action varies as $|Z-Z_+|^{-\frac{3}{2}}$, and the wave amplitude as $|Z-Z_+|^{-\frac{1}{4}}$. The valve effect disappears when $2\Omega_H m$ is zero. It should be noted that in this case

$$\delta n/n \approx \pm \epsilon\pi S\{(N^2-\omega^{*2})\,[\omega^{*2}-(2\Omega_V)^2]\,(l^2+m^2)\}^{-\frac{1}{2}}, \tag{4.27}$$

and so the WKBJ approximation fails as the critical level is approached.

Finally, if $2\Omega_H$ and $2\Omega_V$ are both zero, the valve effect again disappears, and

$$n = \pm\{(N^2-\omega^{*2})\,(l^2+m^2)/\omega^{*2}\}^{\frac{1}{2}}, \tag{4.28}$$

$$W_g = -N^2 n(l^2+m^2)/\omega^*(l^2+m^2+n^2)^2. \tag{4.29}$$

There is now a single critical level at $\omega^* = 0$, both roots for n become infinite as $|Z-Z_0|^{-1}$ (here $Z_0 = \epsilon z_0$), W tends to zero as $|Z-Z_0|^{-2}$, the time $T \rightarrow \infty$ as $|Z-Z_0|^{-1}$ and so the wave packet is captured (cf. Bretherton 1966; Grimshaw 1972, 1974*a*). The wave action $\mathscr{F}$ varies as $|Z-Z_0|^{-2}$, and the wave amplitude as $|Z-Z_0|^{-\frac{1}{2}}$. Here

$$\delta n/n \approx \pm\epsilon 2\pi S/N(l^2+m^2)^{\frac{1}{2}}, \tag{4.30}$$

and the WKBJ approximation retains its validity.

Appendix. Derivation of WKBJ approximation

The basis for the WKBJ approximation is that the length scale L of the waves should be much smaller than that associated with the mean flow, namely g/N_1^2; their ratio is ϵ [see (2.1)], which is the crucial small parameter. In §§2 and 3, ϵ was also a small parameter; however in those sections it measured only the magnitude of the mean density stratification, and no assumption was made about the length scale of the mean density profile. Let

$$\mathbf{X} = \epsilon\mathbf{x}, \quad T = \epsilon t. \tag{A 1}$$

Then it will be assumed that the mean flow variables $\mathbf{V}$, R and Q are functions of $\mathbf{X}$ and T, where $\mathbf{x}$ and t are non-dimensional co-ordinates based on length scales L and N_1^{-1} respectively (cf. §2). The mean pressure is now $\epsilon^{-1}Q$, the extra ϵ^{-1} factor being necessary to ensure that the mean pressure gradient is of the same order as the mean density. The equations for the mean flow are therefore

$$\left.\begin{aligned} &\epsilon\mathscr{D}\mathbf{V}/\mathscr{D}T + 2\mathbf{\Omega}\times\mathbf{V} + \epsilon^{-1}R^{-1}\nabla Q + \epsilon^{-1}\mathbf{k} = 0, \\ &\frac{\mathscr{D}R}{\mathscr{D}T} - \frac{F}{\epsilon C^2}\frac{\mathscr{D}Q}{\mathscr{D}T} = 0, \quad \frac{\mathscr{D}R}{\mathscr{D}T} + R\nabla\,.\,\mathbf{V} = 0, \\ \text{where}\qquad &\mathscr{D}/\mathscr{D}T \equiv \partial/\partial T + \mathbf{V}\,.\,\nabla. \end{aligned}\right\} \tag{A 2}$$

Here all differentiations are with respect to $\mathbf{X}$ and T. To $O(\epsilon^2)$, the mean flow is hydrostatic and geostrophic. The horizontal gradients of Q and R are $O(\epsilon)$ smaller than the vertical gradients. If Q and R are time independent to $O(\epsilon)$, then $\mathbf{V}$ is horizontal to $O(\epsilon)$. However, as pointed out by Garrett (1968), if Q and R are unsteady (to leading order in ϵ), then there will be a leading-order vertical velocity W, which will depend only on Z and T to leading order in ϵ. Although this case is rather artificial, it will be included below. If desired, β-plane effects may also be included, in which case $2\Omega_Y$ will be a linear function of Y (Grimshaw 1974*b*).

To obtain the perturbed equations, let [cf. (2.15)]

$$\mathbf{u} = \mathbf{V}+\hat{\mathbf{v}}, \quad p = \epsilon^{-1}Q+\epsilon\hat{q}, \quad \rho = R(1+\epsilon\hat{r}), \tag{A 3}$$

substitute into (2.3)–(2.5) and linearize in the perturbed variables $\hat{\mathbf{v}}$, $\hat{q}$ and $\hat{r}$. The result is

$$\nabla.\hat{\mathbf{v}}-(F/C^2)\,\hat{w} = O(\epsilon^2), \tag{A 4}$$

$$D\hat{\mathbf{v}}/Dt+2\mathbf{\Omega}\times\hat{\mathbf{v}}+R^{-1}\nabla\hat{q}+\hat{r}\mathbf{k}+\epsilon\hat{\mathbf{v}}.\nabla\mathbf{V}+\epsilon\hat{r}2\mathbf{\Omega}\times\mathbf{V} = O(\epsilon^2), \tag{A 5}$$

$$\frac{D\hat{r}}{Dt}-N^2\hat{w}+R^{-1}\hat{\mathbf{v}}_H.\nabla_H R-\frac{F}{C^2}\left(\frac{D\hat{q}}{Dt}-\hat{\mathbf{v}}.2\mathbf{\Omega}\times\mathbf{V}\right)-F\hat{r}\frac{\mathscr{D}Q}{\mathscr{D}T}\overline{\left\{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho c^2}\right)\right\}} = O(\epsilon^2), \tag{A 6}$$

where

$$D/Dt \equiv \partial/\partial t+\mathbf{V}.\nabla. \tag{A 7}$$

Here a subscript H denotes a horizontal component, and all differentiations of $\hat{\mathbf{v}}$, $\hat{r}$ or $\hat{q}$ are with respect to $\mathbf{x}$ and t, but differentiations of $\mathbf{V}$, R or Q are with respect to $\mathbf{X}$ and T. In the last term in (A 6), the overbar denotes evaluation at (Q,R) and the differentiation of $(\rho c^2)^{-1}$ is with respect to ρ at constant pressure p; for an ideal gas this term is zero; also $\mathscr{D}Q/\mathscr{D}T$ is $O(\epsilon)$, unless Q is unsteady to leading order in ϵ, and W is $O(1)$. Note that the Brunt–Väisälä frequency (2.12) is now given by

$$N^2 = -\{R_Z/R+F/\epsilon C^2\}. \tag{A 8}$$

In the absence of rotation, Bretherton (1966) has obtained WKBJ solutions for the system (A 4)–(A 6) when $\mathbf{V}$ is a function of Z only. Frankignoul (1970, 1972) has used WKBJ methods including the effects of rotation, but for a special shear flow and variations in T only. Lewis (1965) has developed WKBJ procedures for general first-order systems governing dispersive waves. Following his procedure, let

$$\hat{\mathbf{v}} = \mathrm{Re}\,(\mathbf{v}'e^{i\theta}), \quad \hat{q} = \mathrm{Re}\,(q'e^{i\theta}), \quad \hat{r} = \mathrm{Re}\,(r'e^{i\theta}), \tag{A 9}$$

where

$$\theta = \epsilon^{-1}\Theta(\mathbf{X},T). \tag{A 10}$$

Here $\mathbf{v}'$, q' and r' are functions of $\mathbf{X}$ and T only. Θ is the phase function, and the local frequency ω and local wavenumber $\boldsymbol{\kappa}$ are given by

$$\omega = -\Theta_T, \quad \boldsymbol{\kappa} = \nabla\Theta. \tag{A 11}$$

Substitution into (A 5)–(A 7) gives

$$\left.\begin{aligned} i\boldsymbol{\kappa}.\mathbf{v}' &= \epsilon I,\\ -i\omega^*\mathbf{v}'+2\mathbf{\Omega}\times\mathbf{v}'+i\boldsymbol{\kappa}R^{-1}q'+r'\mathbf{k} &= \epsilon\mathbf{M},\\ -i\omega^*r'-N^2w' &= \epsilon P,\end{aligned}\right\} \tag{A 12}$$

where

$$\left.\begin{aligned}\omega^* &= \omega-\boldsymbol{\kappa}.\mathbf{V},\\ I &= -\nabla.\mathbf{v}'+(F/\epsilon C^2)\,w'+O(\epsilon),\\ \mathbf{M} &= -\mathscr{D}\mathbf{v}'/\mathscr{D}T-R^{-1}\nabla q'-\mathbf{v}'.\nabla\mathbf{V}-r'2\boldsymbol{\Omega}\times\mathbf{V}+O(\epsilon),\\ P &= -\frac{\mathscr{D}r'}{\mathscr{D}T}-\epsilon^{-1}R^{-1}\,\mathbf{v}'_H.\nabla_H R+\frac{F}{\epsilon C^2}(-i\omega^*q'-\mathbf{v}'.2\boldsymbol{\Omega}\times\mathbf{V})\\ &\qquad -\frac{Fr'}{\epsilon}R\frac{\mathscr{D}Q}{\mathscr{D}T}\overline{\left\{\frac{\partial}{\partial\rho}\left(\frac{1}{\rho c^2}\right)\right\}}+O(\epsilon).\end{aligned}\right\}\qquad \text{(A 13)}$$

From now on, all differentiations are with respect to $\mathbf{X}$ and T. Note that $\nabla_H R$ is $O(\epsilon)$, so that I, $\mathbf{M}$ and P are all $O(1)$ and the right-hand sides of (A 12) are $O(\epsilon)$.

To leading order in ϵ, (A 12) is a set of linear homogeneous equations, and the condition that there be a solution is

$$\kappa^2\omega^{*2} = (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2+N^2\kappa_H^2, \qquad \text{(A 14)}$$

where κ is the magnitude of $\boldsymbol{\kappa}$ (i.e. $(l^2+m^2+n^2)^{\frac{1}{2}}$ if $\boldsymbol{\kappa} = (l,m,n)$) and κ_H is the magnitude of $\boldsymbol{\kappa}_H$, the horizontal component of $\boldsymbol{\kappa}$ (i.e. $(l^2+m^2)^{\frac{1}{2}}$). This is just the dispersion relation for plane internal gravity waves in a rotating fluid (Phillips 1966, p. 193). Here (A 14) is a partial differential equation for the phase Θ by virtue of (A 11). The corresponding wave is

$$\left.\begin{aligned}r' &= aN^2,\\ \mathbf{v}' &= -\frac{i\omega^*a}{\kappa_H^2}(\mathbf{k}\kappa^2-n\boldsymbol{\kappa})-\frac{a(2\boldsymbol{\Omega}.\boldsymbol{\kappa})}{\kappa_H^2}(\boldsymbol{\kappa}\times\mathbf{k}),\\ R^{-1}q' &= inr'/\kappa^2-i\mathbf{v}'.2\boldsymbol{\Omega}\times\boldsymbol{\kappa}/\kappa^2,\end{aligned}\right\}\qquad \text{(A 15)}$$

where $a(\mathbf{X},T)$ is an undetermined amplitude at this stage.

To obtain an equation for the amplitude, it is convenient to introduce the 5-vectors

$$\mathbf{U} = \begin{bmatrix}\mathbf{v}'\\ r'\\ R^{-1}q'\end{bmatrix},\quad \mathscr{M} = \begin{bmatrix}\mathbf{M}\\ P/N^2\\ I\end{bmatrix}, \qquad \text{(A 16)}$$

and write (A 12) in the form

$$\mathbf{A}\mathbf{U} = \epsilon\mathscr{M}, \qquad \text{(A 17)}$$

where $\mathbf{A}$ is the 5×5 matrix

$$\mathbf{A} = \begin{bmatrix}-i\omega^* & -2\Omega_V & 2\Omega_H & 0 & il\\ 2\Omega_V & -i\omega^* & 0 & 0 & im\\ -2\Omega_H & 0 & -i\omega^* & 1 & in\\ 0 & 0 & -1 & -i\omega^*/N^2 & 0\\ il & im & in & 0 & 0\end{bmatrix}. \qquad \text{(A 18)}$$

Note that $\mathbf{A}$ is anti-Hermitian (i.e. $\bar{\mathbf{A}}^{\mathrm{T}} = -\mathbf{A}$), and singular because of (A 14). The condition that the system of linear equations has a non-trivial solution for $\mathbf{U}$ is that $\mathscr{M}$ should be orthogonal to the null vectors of $\bar{\mathbf{A}}^{\mathrm{T}}$; since $\mathbf{A}$ is anti-Hermitian, this implies that $\mathscr{M}$ should be orthogonal to the null vectors of $\mathbf{A}$:

$$\bar{\mathbf{U}}_0^{\mathrm{T}}\mathscr{M} = 0, \quad \text{where} \quad \mathbf{A}\mathbf{U}_0 = 0. \qquad \text{(A 19)}$$

This condition will be applied when $\mathbf{U}_0$ is given by (A 15), and ω^* is a solution of (A 14). To leading order in ϵ, $\mathscr{M}$ can be evaluated with $\mathbf{U} = \mathbf{U}_0$ (i.e. $\mathbf{v}'$, r' and q' given by (A 15)). Then (A 19) is an equation for the amplitude a, and reverting to the previous notation (A 19) is

$$\bar{\mathbf{v}}'.\mathbf{M}+\bar{r}'P/N^2+R^{-1}\bar{q}'I = 0. \tag{A 20}$$

Substituting (A 13) into (A 20), it follows, after considerable manipulation, that

$$\begin{aligned}\bar{\mathbf{v}}'\frac{\mathscr{D}\mathbf{v}'}{\mathscr{D}T}+\frac{1}{N^2}\bar{r}'\frac{\mathscr{D}r'}{\mathscr{D}T}+\frac{1}{R}\{\bar{\mathbf{v}}'.\nabla q'+\bar{q}'\nabla.\mathbf{v}'\}&\\ +\boldsymbol{\kappa}.(\bar{q}'\mathbf{v}'.\nabla\mathbf{V})/R\omega^*-\frac{|r'|^2}{N^4}\frac{\mathscr{D}N^2}{\mathscr{D}T}-\frac{(\mathbf{V}\times\bar{\mathbf{v}}').(\mathbf{v}'.\nabla 2\boldsymbol{\Omega})}{i\omega^*}&\\ +\frac{RZi\omega^*|r'|^2}{\epsilon N^4R}-\frac{\bar{\mathbf{v}}'.(\mathbf{v}'.\nabla(\nabla Q))}{\epsilon Ri\omega^*}&\\ +\frac{F}{\epsilon C^2}\left\{\frac{\bar{r}'}{N^2}\mathbf{v}'.2\boldsymbol{\Omega}\times\mathbf{V}-\frac{r'}{N^2}\bar{\mathbf{v}}'.2\boldsymbol{\Omega}\times\mathbf{V}+\bar{w}'q'-w'\bar{q}'\right\}&= 0.\end{aligned} \tag{A 21}$$

Taking the real part of (A 21) it follows that

$$\begin{aligned}\frac{\mathscr{D}}{\mathscr{D}T}\left(\tfrac{1}{2}|\mathbf{v}'|^2+\frac{1}{2}\frac{|r'|^2}{N^2}\right)+\frac{1}{R}\nabla.(\operatorname{Re}(q'\bar{\mathbf{v}}'))&\\ +\frac{\boldsymbol{\kappa}}{R\omega^*}.\{\operatorname{Re}(q'\bar{\mathbf{v}}').\nabla\mathbf{V}\}-\frac{|r'|^2}{2N^4}\frac{\mathscr{D}N^2}{\mathscr{D}T}-\frac{\omega^*|a|^2}{\kappa_H^2}(2\boldsymbol{\Omega}.\boldsymbol{\kappa})\boldsymbol{\kappa}.\frac{\mathscr{D}2\boldsymbol{\Omega}}{\mathscr{D}T}&= 0.\end{aligned} \tag{A 22}$$

The wave energy density $\mathscr{E}$ is the average (with respect to θ) of $\frac{1}{2}R|\hat{\mathbf{v}}|^2+\frac{1}{2}R|\hat{r}|^2/N^2$; that is,

$$\mathscr{E} = \tfrac{1}{4}R|\mathbf{v}'|^2+\tfrac{1}{4}R|r'|^2/N^2,$$

or

$$\mathscr{E} = \tfrac{1}{2}R\omega^{*2}\kappa^2|a|^2/\kappa_H^2. \tag{A 23}$$

A direct calculation from (A 15) shows that

$$\operatorname{Re}(q'\bar{\mathbf{v}}') = 2\mathscr{E}\mathbf{c}, \tag{A 24}$$

where $\mathbf{c}$ is the group velocity, i.e. $\mathbf{c} = \nabla_\kappa\omega^*$, and is given by

$$\omega^*\kappa^2\mathbf{c} = -N^2n(\mathbf{k}-n\boldsymbol{\kappa}/\kappa^2)+(2\boldsymbol{\Omega}.\boldsymbol{\kappa})(2\boldsymbol{\Omega}-(2\boldsymbol{\Omega}.\boldsymbol{\kappa})\boldsymbol{\kappa}/\kappa^2). \tag{A 25}$$

Also, it may be shown that

$$\frac{\mathscr{D}\omega^*}{\mathscr{D}T}+\mathbf{c}.\nabla\omega^* = -\boldsymbol{\kappa}.(\mathbf{c}.\nabla\mathbf{V})+\frac{1}{2}\frac{\kappa_H^2}{\kappa^2}\frac{1}{\omega^*}\frac{\mathscr{D}N^2}{\mathscr{D}T}+\frac{2\boldsymbol{\Omega}.\boldsymbol{\kappa}}{\kappa^2\omega^*}\boldsymbol{\kappa}.\frac{\mathscr{D}2\boldsymbol{\Omega}}{\mathscr{D}T}. \tag{A 26}$$

Substitution of (A 23)–(A 26) into (A 22) shows that

$$\frac{\mathscr{D}\mathscr{E}}{\mathscr{D}T}-\frac{\mathscr{E}}{R}\frac{\mathscr{D}R}{\mathscr{D}T}+\nabla.(\mathscr{E}\mathbf{c}) = \frac{\mathscr{E}}{\omega^*}\left(\frac{\mathscr{D}\omega^*}{\mathscr{D}T}+\mathbf{c}.\nabla\omega^*\right). \tag{A 27}$$

Finally, using (A 2) to eliminate R gives the result

$$\partial\mathscr{F}/\partial T+\nabla.(\mathscr{F}(\mathbf{c}+\mathbf{V})) = 0, \tag{A 28}$$

where

$$\mathscr{F} = \mathscr{E}/\omega^*.$$

Here $\mathscr{F}$ is the wave action. The conservation of wave action in the absence of rotation was established by Bretherton (1966) (q.v. also Grimshaw 1972, 1974*a*), and in a variety of other physical contexts by Bretherton & Garrett (1968). Equation (A 28) shows that the amplitude of the wave, measured by the wave action $\mathscr{F}$, moves with the group velocity **c**.

Equation (A 28) is an equation for $|a|$. The imaginary part of (A 21) leads to an equation for $\arg a$ of the form

$$(\mathscr{D}/\mathscr{D}T+\mathbf{c}.\nabla)\arg a+\ldots = 0, \tag{A 29}$$

where the omitted terms do not involve $\arg a$, and have not been displayed as they are rather involved.

REFERENCES

ACHESON, D. J. 1972 *J. Fluid Mech.* **53**, 401–415.

ACHESON, D. J. 1973 *J. Fluid Mech.* **58**, 27–37.

BOOKER, J. R. & BRETHERTON, F. P. 1967 *J. Fluid Mech.* **27**, 513–539.

BRETHERTON, F. P. 1966 *Quart. J. Roy. Met. Soc.* **92**, 466–480.

BRETHERTON, F. P. & GARRETT, C. J. R. 1968 *Proc. Roy. Soc.* A **302**, 529–554.

FRANKIGNOUL, C. J. 1970 *Tellus*, **22**, 194–203.

FRANKIGNOUL, C. J. 1972 *Mém. Soc. Roy. Sci. Liege* **2** (6), 51–58.

GARRETT, C. J. R. 1968 *J. Fluid Mech.* **34**, 711–720.

GRIMSHAW, R. 1972 *J. Fluid Mech.* **54**, 193–207.

GRIMSHAW, R. 1974*a* *Geophys. Fluid Dyn.* **6**, 131–148.

GRIMSHAW, R. 1974*b* *School Math. Sci., University of Melbourne, Resp. Rep.* no. 20.

JONES, W. L. 1967 *J. Fluid Mech.* **30**, 439–448.

LEWIS, R. M. 1965 *Arch. Rat. Mech. Anal.* **20**, 191–250.

McKENZIE, J. F. 1973 *J. Fluid Mech.* **58**, 709–726.

PHILLIPS, O. M. 1966 *The Dynamics of the Upper Ocean.* Cambridge University Press.

RUDRAIAH, N. & VENKATACHALAPPA, M. 1972*a* *J. Fluid Mech.* **52**, 193–206.

RUDRAIAH, N. & VENKATACHALAPPA, M. 1972*b* *J. Fluid Mech.* **54**, 217–240.